- Second derivative test (cont.) ← This only works (as stated below) for $f(x,y)$

- Let $f(x,y)$ be differentiable @ $\vec{p}$ and $\vec{p}$ is a CP.

① If $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$, then $\vec{p}$ is a local min point of f.

② If $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(\vec{p}) - (f_{xy}(\vec{p}))^2 > 0$, then $\vec{p}$ is a local max point of f.

③ If $D(\vec{p}) = f_{xx}(\vec{p}) f_{yy}(p) - (f_{xy}(\vec{p}))^2 < 0$, then $\vec{p}$ is a saddle point of f.

N.B: ① If $D(\vec{p}) = 0$, 2^{nd} derivative test gives no info.
② you can also use f_{yy} in place of f_{xx} for ① and ②

Ex] classify via 2^{nd} derivative test all CPs of $f(x,y) = x^2 + xy + y^2 + y$.
① find critical points ② use 2^{nd} derivative test to say as much as you can about them.

Sol] ① find $\nabla f = \langle 2x+y,\ x+2y+1 \rangle$

$\therefore \nabla f = 0$ iff $\langle 2x+y,\ x+2y+1 \rangle = 0 \rightarrow \begin{matrix} 2x+y=0 \\ x+2y+1=0 \end{matrix} \rightarrow \begin{matrix} y=-2x \\ x+2(-2x)+1=0 \end{matrix}$

$\rightarrow \begin{matrix} -3x+1=0 \\ y=-2x \end{matrix} \rightarrow \begin{matrix} x=1/3 \\ y=-2/3 \end{matrix}$

$\therefore f$ has a unique CP at $(\frac{1}{3}, \frac{-2}{3})$

via 2^{nd} derivative test:

$f_{xx}=2$, $f_{yy}=2$, $f_{xy}=1$

$\therefore D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3$

at $p = (\frac{1}{3}, \frac{-2}{3})$, $D_p = 3 > 0$ and $f_{xx}(\frac{1}{3}, \frac{-2}{3}) = 2 > 0$

So $(\frac{1}{3}, \frac{-2}{3})$ is a local min point with local min value with $f(\frac{1}{3}, \frac{-2}{3}) = (\frac{1}{3})^2 + (\frac{1}{3})(\frac{-2}{3}) + (\frac{-2}{3})^2 + \frac{-2}{3} = -1/3$

[ex] classify CPs of $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9y$

Sol] $\nabla f = \langle 3x^2 - 6x, \; 3y^2 - 6y - 9 \rangle = 3\langle x^2 - 2x, \; y^2 - 2y - 3 \rangle$

$\nabla f = 3\langle x(x-2), \; (y-3)(y+1) \rangle$

$\nabla f = 0$ iff $\begin{cases} x(x-2) = 0 \\ (y-3)(y+1) = 0 \end{cases}$ iff $\begin{cases} x = 0 \text{ or } x = 2 \\ y = 3 \text{ or } y = -1 \end{cases}$

	$x=0$	$x=2$
$y=3$	$(0,3)$	$(2,3)$
$y=-1$	$(0,-1)$	$(2,-1)$

analyze these points via 2nd Derivative test

all 4 of these points are CPs.

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

$f_{xx} = 6x - 6 = 6(x-1)$

$f_{yy} = 6y - 6 = 6(y-1)$

$f_{xy} = f_{xy} = 0$

$D = [6(x-1)] \cdot [6(y-1)] + 0^2$

$D = 6^2(x-1)(y-1)$

For $p = (0,3)$:

$D_p = 6^2(0-1)(3-1) = 6^2(-2) = -72$

So $p = (0,3)$ is a saddle point of f.

For $p = (0,-1)$:

$D_p = 6^2(0-1)(-1-1) > 0$

$f_{xx}(0,-1) = 6(0-1) < 0$

So $p(0,-1)$ is a local max of f w/ local max value

$f(0,-1) = 5$ ~~[illegible]~~ don't know

For $p = (2,3)$:

$D_p = 6^2(2-1)(3-1) > 0$

$f_{xx} = 6(2-1) > 0$

$\therefore p(2,3)$ is a local min of f with loal min value

$f(2,3) = -3$

For $p = (2,-1)$

$D_p = 6^2(2-1)(-1-1) < 0$

$\therefore p = (2,-1)$ is a saddle point

[Ex] classify CPs of $f(x,y) = xy + e^{-xy}$

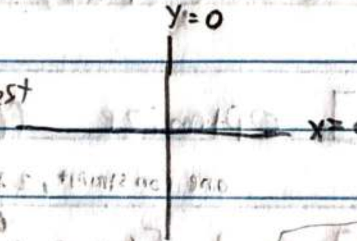
Sol] $\nabla f = \langle y - ye^{-xy}, x - xe^{-xy} \rangle = \langle y(1-e^{-xy}), x(1-e^{-xy}) \rangle$

$\nabla f = 0$ iff $\begin{cases} y(1-e^{-xy}) = 0 \\ x(1-e^{-xy}) = 0 \end{cases}$ iff $\begin{cases} y=0 \text{ or } 1-e^{-xy}=0 \\ x=0 \text{ or } 1-e^{-xy}=0 \end{cases}$

$1-e^{-xy}=0$ iff $e^{-xy}=1$ iff $-xy=0$ iff $x=0$ or $y=0$

$\therefore \nabla f = 0$ iff $\begin{cases} y=0 \text{ or } (y=0 \text{ or } x=0) \\ x=0 \text{ or } (y=0 \text{ or } x=0) \end{cases}$ iff $\begin{cases} x=0 \text{ or } y=0 \\ y=0 \text{ or } x=0 \end{cases}$

$\therefore \nabla f = 0$ iff $x=0$ or $y=0$ $\rightarrow$



Now: Analyze CPs using 2nd d test

$f_{xx} = y^2 e^{-xy}$

$f_{yy} = x^2 e^{-xy}$

$f_{xy} = 1 - (e^{-xy} + y(-xe^{-xy})) = 1 - e^{-xy}(1-xy)$

$\therefore D(x,y) = (y^2 e^{-xy})(x^2 e^{-xy}) - (1-e^{-xy}(1-xy))^2$

$= (xy)^2 (e^{-2xy}) - (1-e^{-x}(1-xy))^2$

either $x=0$ or $y=0$ iff $xy=0$

$\therefore$ Every CP satisfies

$D(x,y) = 0^2 e^{-2(0)} - (1-e^{-0}(1-0))^2$

$= 0 - (1-1) = 0$

$\therefore$ 2nd Derivative test is inconclusive on all of thes CPs

- La grange multipliers:

Goal: build a method to systematically solve constrained optimization problem.

Problem: You need to: $\begin{cases} \text{optimize } f(\vec{x}) \\ \text{Subject to } g_1(\vec{x}) = g_2(\vec{x}) = \ldots = g_k(\vec{x}) = 0 \end{cases}$

- Observation: If we want extreme values of F on a level set $F(\vec{x}) = c$, what we really want are CPs of F because $\nabla F = \nabla c = 0$

$\longrightarrow$

Consider F derived from the problem:
$F(\vec{x},\lambda_1,\lambda_2 \dots \lambda_k) \equiv f(\vec{x}) - \lambda_1 g_1(\vec{x}) - \lambda_2 g_2(\vec{x}) \dots - \lambda_k g_k(\vec{x})$
Now (because of level set considerations
see the video), global solutions to the problem
occur only at CPs of F.
$\therefore$ we now need to solve $\nabla f = 0$ and find
absolute max/min values.

[Ex] optimize $f(x,y) = xe^y$ along $x^2+y^2=2$

one constraint, 1 λ

Sol] need $F(x,y,\lambda)$. Note: $x^2+y^2=2$ iff $x^2+y^2-2=0$
So we use $g(x,y) = x^2+y^2-2$

$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$
$= xe^y - \lambda(x^2+y^2-2)$

$\therefore \nabla F = \langle e^y - \lambda 2x,\ xe^y - \lambda 2y,\ -(x^2+y^2-2)\rangle$

$\nabla F = 0$ iff $\begin{cases} e^y - \lambda 2x = 0 \\ xe^y - \lambda 2y = 0 \\ -(x^2+y^2-2)=0 \end{cases}$ iff $\begin{cases} 2\lambda x = e^y & (1) \\ 2\lambda y = xe^y & (2) \\ x^2+y^2 = 2 & (3) \end{cases}$

by (1) $\lambda \neq 0$ because $2\lambda x = e^y$ and e^y is never 0 for real inputs.
by (2) and plugging in (1) $2\lambda y = xe^y = x(2\lambda x) = 2\lambda x^2 \rightarrow \underline{y = x^2}$
by (3) $x^2+y^2=2 \rightarrow x^2+(x^2)^2-2=0 \rightarrow (x^2+2)(x^2-1)=0$
$\therefore (x^2+2)(x+1)(x-1)=0$ so $x = \pm 1$ for all CPs of F
if $x=1$: $y=1^2$ and $\lambda = \frac{e^y}{2x} = \frac{e^1}{2(1)} = \frac{e}{2}$ $\therefore (1,1)$ is a possible
extreme point of f w/ corresponding value
$f(1,1) = 1e^1 = e$

if $x=-1$; $y=(-1)^2=1$ and $\lambda = \frac{e^y}{2x} = \frac{e^1}{2(-1)} = \frac{-e}{2}$

$\therefore (-1,1)$ is a possible extreme point

of w/ value $f(-1,1)=-e$

$\therefore$ e is the global max and $-e$ is the global min

for f subject to $x^2+y^2=2$